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Generalized geometry, calibrations and supersymmetry in diverse dimensions

Dieter Lüst[◇♣], **Peter Patalong**[♣] **and Dimitrios Tsimpis**[◇]

◇ *Arnold-Sommerfeld-Center for Theoretical Physics*
Department für Physik, Ludwig-Maximilians-Universität München
Theresienstraße 37, 80333 München, Germany

♣ *Max-Planck-Institut für Physik – Theorie*
Föhringer Ring 6, 80805 München, Germany

E-mail:
luest@mppmu.mpg.de, luest@lmu.de , peter.patalong@physik.uni-muenchen.de &
dimitrios.tsimpis@lmu.de

Abstract: We consider type II backgrounds of the form $\mathbb{R}^{1,d-1} \times \mathcal{M}_{10-d}$ for even d, preserving $2^{d/2}$ real supercharges; for $d = 4, 6, 8$ this is minimal supersymmetry in d dimensions, while for $d = 2$ it is $\mathcal{N} = (2, 0)$ supersymmetry in two dimensions. For $d = 6$ we prove, by explicitly solving the Killing-spinor equations, that there is a one-to-one correspondence between background supersymmetry equations in pure-spinor form and D-brane generalized calibrations; this correspondence had been known to hold in the $d = 4$ case. Assuming the correspondence to hold for all d, we list the calibration forms for all admissible D-branes, as well as the background supersymmetry equations in pure-spinor form. We find a number of general features, including the following: The pattern of codimensions at which each calibration form appears exhibits a (mod 4) periodicity. In all cases one of the pure-spinor equations implies that the internal manifold is generalized Calabi-Yau. Our results are manifestly invariant under generalized mirror symmetry.

Keywords: Generalized geometry, supersymmetric backgrounds, flux compactifications.

Contents

1. Introduction

Generalized geometry [1, 2] (see [3] for a review) provides a natural mathematical framework for the description of type II flux backgrounds. It has lead to important insights into many recent developments, such as explicit supersymmetric solutions, effective actions, sigma models, as well as supersymmetry breaking and non-geometry. In the language of generalized geometry the supersymmetry conditions for a background of the form $\mathbb{R}^{1,3} \times \mathcal{M}_6$

are expressed as a set of first-order differential equations for two complex pure spinors of Cliff$(6,6)$ [4]; the latter can be thought of equivalently as polyforms on $\mathcal{M}_6$.

The close connection between background supersymmetry and calibrated branes [5, 6, 7] has been noted in various different setups [8, 9, 10], and calibrations have a natural interpretation within the context of generalized geometry [11, 12, 13, 14, 15]. For type II backgrounds of the form $\mathbb{R}^{1,3} \times \mathcal{M}_6$ in particular, this connection works as follows [12]: When written in their pure-spinor form, the supersymmetry equations of the background[1] are in one-to-one correspondence with the differential conditions obeyed by the calibration forms of all admissible static, magnetic D-branes in that background. It is natural to expect that this correspondence extends more generally to all type II backgrounds of the form $\mathbb{R}^{1,d-1} \times \mathcal{M}_{10-d}$ which can be described with generalized geometry (as we will see in the following this requirement restricts d to be even).

In the present paper we show that this is indeed the case for minimally-supersymmetric (four complex supercharges), type II backgrounds of the form $\mathbb{R}^{1,5} \times \mathcal{M}_4$. We prove this by a brute-force computation involving the following steps: a) we explicitly give the general solution of the Killing-spinor equations (*i.e.* the supersymmetry conditions) of the background; b) we write down the set of differential equations, in pure-spinor form, obeyed by all admissible static, magnetic, calibrated D-branes in that background; c) we show that the solution of the set of equations in b) is the same as the solution in a).

Based on these results, we conjecture that the one-to-one correspondence between calibrated D-branes and background supersymmetry holds for all (even) d, for backgrounds of the form $\mathbb{R}^{1,d-1} \times \mathcal{M}_{10-d}$ with $2^{d/2-1}$ complex supercharges. This is the amount of supersymmetry parameterized by a complexified Weyl spinor in d dimensions: for $d = 4, 6, 8$ it corresponds to minimal supersymmetry in d dimensions; for $d = 2$ it corresponds to $\mathcal{N} = (2, 0)$ supersymmetry in two dimensions.

Assuming the correspondence to be true allows us to deduce the supersymmetry equations for the background in pure-spinor form for the remaining two non-trivial cases corresponding to $d = 2, 8$, by performing the much easier task of computing the calibration forms of all admissible D-branes in that background. The summary of the supersymmetry equations for all $\mathbb{R}^{1,d-1} \times \mathcal{M}_{10-d}$ backgrounds with $2^{d/2-1}$ complex supercharges is given in eq. (1.1) below;

[1] In this paper we assume that the internal manifold admits a pair of compatible, globally-defined, nowhere-vanishing pure spinors; as will be reviewed in the following, this implies the reduction of the structure group of the direct sum of the tangent and cotangent bundles of the internal manifold to $SU(k) \times SU(k)$, where $k := (10 - d)/2$. We will moreover assume that the background admits calibrated D-branes; this implies a certain restriction on the norm of the Killing spinors of the background.

$$\boxed{\begin{array}{lll} \mathrm{d}_H\left(e^{(d-4r)A-\Phi}\mathrm{Re}\Psi_1\right)=\delta_{r,0}F^{\mathrm{el}} & & d-4r\geq 1 \\ \mathrm{d}_H\left(e^{(d-2-4r)A-\Phi}\mathrm{Im}\Psi_1\right)=0 & & d-2-4r\geq 1 \\ \mathrm{d}_H\left(e^{[\frac{1}{2}(d+2)-4r]A-\Phi}\Psi_2\right)=0 & & \frac{1}{2}(d+2)-4r\geq 1\ , \end{array}} \tag{1.1}$$

where $r\in\mathbb{N}$; our notation and conventions for the fluxes are described in detail in section 3. As will be explained in detail in the following sections, each of these pure-spinor equations can be thought of as the differential condition obeyed by the calibration form for a D-brane of the corresponding codimension – the latter being equal to d minus the coefficient of A in the exponential. For each d the different calibration forms and their corresponding codimensions are given in table 1.

		External spacetime dimension d			
		2	**4**	**6**	**8**
Codimension	**0**	$e^{2A-\Phi}\mathrm{Re}\Psi_1$, $e^{2A-\Phi}\Psi_2$	$e^{4A-\Phi}\mathrm{Re}\Psi_1$	$e^{6A-\Phi}\mathrm{Re}\Psi_1$	$e^{8A-\Phi}\mathrm{Re}\Psi_1$
	1		$e^{3A-\Phi}\Psi_2$		
	2		$e^{2A-\Phi}\mathrm{Im}\Psi_1$	$e^{4A-\Phi}\mathrm{Im}\Psi_1$, $e^{4A-\Phi}\Psi_2$	$e^{6A-\Phi}\mathrm{Im}\Psi_1$
	3				$e^{5A-\Phi}\Psi_2$
	4			$e^{2A-\Phi}\mathrm{Re}\Psi_1$	$e^{4A-\Phi}\mathrm{Re}\Psi_1$
	5				
	6				$e^{2A-\Phi}\mathrm{Im}\Psi_1$
	7				$e^{A-\Phi}\Psi_2$

Table 1: The 'periodic table' of calibration forms for all d, each of them corresponding to a background supersymmetry pure-spinor equation. This one-to-one correspondence had been known to hold in the $d=4$ case, and in the present paper is also shown to hold in the $d=6$ case. Based on these results we conjecture it to hold for the $d=2,8$ cases as well.

Having the complete 'periodic table' of pure-spinor supersymmetry equations and their one-to-one correspondence with calibrations, allows one to identify a number of general patterns:

- The 'critical dimension' where there are four (real) pure-spinor equations for two (complex) pure spinors $\Psi_{1,2}$ is $d=4$. For $d>4$ there are more equations; for $d<4$ there are fewer equations.
- One of the equations is always the (twisted) closure of a pure spinor (Ψ_2 with an

appropriate warp factor). As will be reviewed in the following, this implies that the internal manifold is Generalized Calabi-Yau.[2]

- $\mathrm{Re}\Psi_1$ is associated with calibrations of codimension 0 mod 4, while $\mathrm{Im}\Psi_1$ is associated with calibrations of codimension 2 mod 4; Ψ_2 is associated with calibrations of codimension $(d/2-1)$ mod 4.

- Eqs. (1.1) as well as the generalized calibrations of table 1 take the same form in both IIA and IIB, hence our results are manifestly invariant under 'generalized mirror symmetry'.[3]

Finally, as an illustration of the pure-spinor formalism in the $d=6$ case, we construct a type IIB warped K3 solution with spacetime-filling D5 branes localized on the K3. In the degenerate limit where K3 is replaced by a T^4 we show that the solution coincides with the one obtained using the 'harmonic superposition rules' for a stack of D5 branes in flat space. We also construct a T-dual IIA warped $S^1\times T^3$ solution with spacetime-filling D6 branes localized on the T^3 and wrapping the S^1.

The remainder of the paper is organized as follows. In section 2 we give a brief introduction to generalized geometry. The type II flux backgrounds which we consider are described in detail in section 3. Section 3.1 contains a review of calibrations in the present context. The different admissible D-brane calibrations for all d are constructed in section 4. We give our conclusions in section 5. In appendix A we list some useful spinor and gamma-matrix identities. Appendix B contains our proof of the one-to-one correspondence between background supersymmetry equations and calibrations in $d=6$. The warped K3 and $S^1\times T^3$ solutions are given in appendix C.

2. Generalized geometry

For completeness we briefly review here the relevant concepts of generalized complex geometry [1, 2]. We refer to *e.g.* the recent review [3] for detailed explanations and references.

Generalized almost complex structures

Generalized complex geometry is an extension of both complex and symplectic geometry, interpolating, in a sense which we will make precise in the following, between these two special cases. Consider an even-dimensional manifold $\mathcal{M}_{2k}$. One can equip the sum of tangent and cotangent bundles $T\oplus T^*$ with a metric of maximally indefinite signature $\mathcal{G}$ (the pairing between vectors and forms), reducing the structure group to $O(2k,2k)$. Imposing in addition the existence of an almost complex structure $\mathcal{I}$ on $T\oplus T^*$ associated

[2]In the limit of vanishing flux the internal manifold reduces to an ordinary Calabi-Yau. Recall that in two real dimensions a Calabi-Yau manifold is a T^2, while in four real dimensions it is a K3 surface.

[3]Generalized mirror symmetry may be thought of as the action of reversing the chirality of the pure spinors $\Psi_{1,2}$ in IIA/IIB. Explicitly, if we define $\Psi_+=\Psi_{2/1}$ and $\Psi_-=\Psi_{1/2}$ in IIA/IIB, then generalized mirror symmetry acts by exchanging $\Psi_+\leftrightarrow\Psi_-$ and IIA $\leftrightarrow$ IIB. We refer to [3] for further discussion.

with the metric $\mathcal{G}$ (*i.e.* such that $\mathcal{G}$ is hermitian with respect to $\mathcal{I}$), further reduces the structure group to $U(k,k)$.

A pair $\mathcal{I}_{1,2}$ of *compatible* almost complex structures on $T \oplus T^*$ (*i.e.* such that they commute and they give rise to a positive definite metric) further reduces the structure group to $U(k) \times U(k)$. The metric on $T \oplus T^*$ associated with the pair $\mathcal{I}_{1,2}$ can be seen to give rise to both a positive definite metric g and a B-field on T.

(Generalized) almost complex structures and pure spinors

Just as there is an equivalence between almost complex structures on T and line bundles of pure Weyl spinors of Cliff$(2k)$,[4] there is an equivalence between almost complex structures on $T \oplus T^*$ and line bundles of pure spinors of Cliff$(2k, 2k)$. Demanding that the line bundle of pure spinors of Cliff$(2k, 2k)$ have a global section, reduces the structure group of $T \oplus T^*$ from $U(k,k)$ (which was accomplished by the existence of a generalized almost complex structure) to $SU(k,k)$.

Spinors on $T \oplus T^*$, bispinors on T, polyforms in $\Lambda^\bullet T^*$

There is a natural action of $T \oplus T^*$ on the bundle $\Lambda^\bullet T^*$ of differential forms on $\mathcal{M}_{2k}$, whereby every vector acts by contraction and every one-form by exterior multiplication. It can easily be seen that this action obeys the Clifford algebra Cliff$(2k, 2k)$ associated with the maximally indefinite metric $\mathcal{G}$ on $T \oplus T^*$. It follows that there is an isomorphism Cliff$(2k, 2k) \approx \mathrm{End}(\Lambda^\bullet T^*)$, which means that spinors on $T \oplus T^*$ can be identified with *polyforms* (*i.e.* sums of forms of different degrees) in $\Lambda^\bullet T^*$.

On the other hand, there is a correspondence between polyforms of $\Lambda^\bullet T^*$ and *bispinors* on T. This correspondence is a canonical isomorphism, up to a choice of the volume form, and is explicitly realized by the *Clifford map*:

$$\psi_\alpha \otimes \widetilde{\chi}_\beta = \frac{1}{2^k} \sum_{p=0}^{2k} \frac{1}{p!} (\widetilde{\chi} \gamma_{m_p \ldots m_1} \psi)\, \gamma_{\alpha\beta}^{m_1 \ldots m_p} \longleftrightarrow \frac{1}{2^k} \sum_{p=0}^{2k} \frac{1}{p!} (\widetilde{\chi} \gamma_{m_p \ldots m_1} \psi)\, e^{m_1} \wedge \cdots \wedge e^{m_p} \ , \quad (2.1)$$

where the first equality is the Fierz identity.

Pairs of compatible pure spinors and $SU(k) \times SU(k)$ structures

It follows from the above discussion that the condition of compatibility of a pair of generalized almost complex structures should be expressible as a condition of compatibility on a pair of (line bundles of) pure spinors of Cliff$(2k, 2k)$ – which, as already mentioned, can alternatively be thought of as either bispinors of Cliff$(2k)$ or, through eq. (2.1), as polyforms. Indeed, the most general form of a pair $\Psi_{1,2}$ of compatible pure spinors of

[4] Recall that pure Weyl spinors may be defined as the spinors which are annihilated by precisely those gamma matrices that are holomorphic (or antiholomorphic, depending on the convention) with respect to an almost complex structure.

Cliff$(2k, 2k)$ is given by:

$$\begin{aligned}\Psi_1 &= \frac{(2i)^k}{|a|^2}\eta_1 \otimes \widetilde{\eta}_2^c \\ \Psi_2 &= \frac{(2i)^k}{|a|^2}\eta_1 \otimes \widetilde{\eta}_2 \ ,\end{aligned} \tag{2.2}$$

where $\eta_{1,2}$ are pure spinors[5] of Cliff$(2k)$. The normalization above is chosen for future convenience, and we have imposed that the background admits calibrated branes, in which case $\eta_{1,2}$ have equal norm: $|a|^2 := \widetilde{\eta}_1\eta_1^c = \widetilde{\eta}_2\eta_2^c$; see appendix A for our spinor conventions.

Provided the pair of pure spinors above is globally defined and nowhere vanishing (in other words: if the corresponding line bundles of pure spinors have nowhere-vanishing global sections), the structure group of $T \oplus T^*$ is further reduced from $U(k) \times U(k)$ (which was accomplished by the existence of a pair of compatible generalized almost complex structures) to $SU(k) \times SU(k)$.

Generalized complex manifolds, and GCY

The correspondence between generalized almost complex structures and pure spinors allows one to express the condition of integrability of a generalized almost complex structure as a certain first-order differential equation for the associated pure spinor, which may then also be called integrable. A manifold $\mathcal{M}_{2k}$ is called *generalized complex* if it admits an integrable pure spinor. It can be shown that if $\mathcal{M}_{2k}$ is generalized complex, it is locally equivalent to $\mathbb{C}^q \times (\mathbb{R}^{2(k-q)}, J)$, with J the standard symplectic structure; thus generalized complex geometry can be said to be an interpolation between complex and symplectic geometries. The integer q is called the *type*, and need not be constant over $\mathcal{M}_{2k}$.

A *generalized Calabi-Yau (GCY)* is a special case of a generalized complex manifold. It is defined as a manifold $\mathcal{M}_{2k}$ on which a pure spinor Ψ exists, obeying the differential condition[6]

$$\mathrm{d}_H\Psi = 0 \ , \tag{2.3}$$

where $\mathrm{d}_H := d + H\wedge$ and $H = dB$ is the field strength of the B field. The presence of the latter should not be too surprising, as we have already mentioned that pairs of compatible pure spinors naturally incorporate a B field.

3. Supersymmetric flux backgrounds

Let us now describe our supergravity setup in more detail.

We consider ten-dimensional type IIA/IIB backgrounds of the form:

$$\mathrm{d}s^2 = e^{2A}\mathrm{d}s^2(\mathbb{R}^{1,d-1}) + \mathrm{d}s^2(\mathcal{M}_{10-d}) \ , \tag{3.1}$$

[5] Note that for $k \leq 3$, Weyl spinors of Cliff$(2k)$ are automatically pure. For the case $k = 4$ one has to impose in addition one complex condition; we will return to this in section 4.4.

[6] This is also sometimes called the 'twisted' Calabi-Yau condition; the pure spinor Ψ is thought of as a polyform in $\Lambda^\bullet T^*$ via the Clifford map (2.1).

where:

$$d = 2, 4, 6, 8 \ . \tag{3.2}$$

The case $d = 10$ is trivial and will not be considered separately. The warp factor A is taken to only depend on the coordinates of the 'internal' Riemannian manifold $\mathcal{M}_{10-d}$.

We assume that not all RR charges are zero; the case with zero RR charges has already been analyzed in [9]. The most general RR charges respecting the Poincaré symmetry of our setup are of the form:[7]

$$F^{\text{tot}} = \text{vol}_d \wedge F^{\text{el}} + F \ , \tag{3.3}$$

where vol_d is the unwarped volume element of $\mathbb{R}^{1,d-1}$, and we are using polyform notation. We denote by F the 'magnetic' RR charges with legs on the intenral space $\mathcal{M}_{10-d}$. The ten-dimensional Hodge duality relates F to the 'electric' RR charges via:

$$F^{\text{el}} = \left(e^{A}\right)^{d} \star_{10-d} \sigma(F) \ , \tag{3.4}$$

where the Hodge star above is with respect to the internal metric, and the involution σ acts by inverting the order of the form indices.

We consider backgrounds preserving $2^{d/2-1}$ complex supercharges. Note that the dimension of a Weyl spinor of $\mathbb{R}^{1,d-1}$ is precisely $\dim(\text{Weyl}_d) = 2^{d/2-1}$, so that the supercharges are parameterized by a complexified[8] Weyl spinor ζ of $\mathbb{R}^{1,d-1}$. More explicitly, the Killing spinors of the ten-dimensional background are given by:

$$\epsilon_i = \zeta \otimes \eta_i + \text{c.c.} \ , \tag{3.5}$$

where $i = 1, 2$, so that $\epsilon_{1,2}$ are ten-dimensional Majorana-Weyl spinors of opposite, the same chirality for IIA, IIB respectively. The spinors $\eta_{1,2}$ are pure Weyl spinors (*cf.* footnote 5) of Cliff$(10-d)$ of opposite, the same chirality for IIA, IIB respectively. The precise form of the complex conjugate on the right hand side of the equation above depends on the dimension d and will be given explicitly in the following.

For $d = 4, 6, 8$ the Killing spinor ansatz given in eq. (3.5) corresponds to minimal supersymmetry in d dimensions; for $d = 2$ it corresponds to $\mathcal{N} = (2, 0)$ supersymmetry in two dimensions.

3.1 Calibrations

The close connection between supersymmetry and calibrations was noted some time ago [5, 6, 7]. More recently, generalized calibrations in flux backgrounds were shown to have a natural interpretation in terms of generalized geometry [11, 12, 14]. In this section we will briefly review the relevant results, referring the reader to [3] or the original literature for further details.

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[7] We follow the 'democratic' supergravity conventions of [16], see appendix A therein.

[8] We use the terminology 'complexified' for a Weyl spinor with complex components. The term 'complex Weyl spinor' is reserved for Weyl spinors whose complex conjugate has opposite chirality.

Consider the energy density $\mathcal{E}(\Sigma, \mathcal{F})$ of a static, magnetic (*i.e.* without electric worldvolume flux) D-brane in our setup, filling q external spacetime dimensions and wrapping a cycle Σ in the internal space (for our purposes it will not be necessary to take higher-order corrections into consideration):

$$\mathcal{E}(\Sigma, \mathcal{F}) = e^{qA-\Phi}\sqrt{det(g+\mathcal{F})} - \delta_{q,d}\left(C^{\text{el}} \wedge e^{\mathcal{F}}\right)_{\Sigma} \ , \tag{3.6}$$

where g is the induced worldvolume metric on Σ, $\mathcal{F}$ is the worldvolume flux: $d\mathcal{F} = H|_{\Sigma}$, and C^{el} is the electric RR flux potential: $\mathrm{d}_H C^{\text{el}} = F^{\text{el}}$, *cf.* eq. (3.3). Note that unless the brane fills all the external spacetime directions, the second term on the right hand side above vanishes. This property of the energy density follows from the form of the ansatz for the RR fields, eq. (3.3), which is such that it preserves the d-dimensional Poincaré invariance of the background.

A polyform ω (defined in the whole of the internal space) is a *generalized calibration form* if, for any cycle Σ, it satisfies the algebraic inequality:

$$\left(\omega \wedge e^{\mathcal{F}}\right)_{\Sigma} \leq \mathrm{d}\sigma e^{qA-\Phi}\sqrt{det(g+\mathcal{F})} \ , \tag{3.7}$$

where σ collectively denotes the coordinates of Σ, together with the differential condition:[9]

$$\mathrm{d}_H \omega = \delta_{q,d} F^{\text{el}} \ . \tag{3.8}$$

A generalized submanifold $(\Sigma, \mathcal{F})$ is called *calibrated by* ω, if it saturates the bound given in eq. (3.7) above.

The upshot of the above discussion is that D-branes wrapping generalized calibrated submanifolds minimize their energy within their (generalized) homology class. Recall that $(\Sigma, \mathcal{F})$, $(\Sigma', \mathcal{F}')$ are in the same generalized homology class if there is a cycle $\widetilde{\Sigma}$ such that $\partial\widetilde{\Sigma} = \Sigma' - \Sigma$ and there exists an extension of the worldvolume flux $\widetilde{\mathcal{F}}$ on $\widetilde{\Sigma}$ such that: $\widetilde{\mathcal{F}}|_{\Sigma} = \mathcal{F}$ and $\widetilde{\mathcal{F}}|_{\Sigma'} = \mathcal{F}'$. Then, if $(\Sigma, \mathcal{F})$ is calibrated by ω we have, using Stokes theorem as well as eqs. (3.6-3.8):

$$\int_{\Sigma'} \mathrm{d}\sigma\, \mathcal{E}(\Sigma', \mathcal{F}') \geq \int \left(\omega - \delta_{q,d}C^{\text{el}}\right)_{\Sigma'} \wedge e^{\mathcal{F}'} = \int \left(\omega - \delta_{q,d}C^{\text{el}}\right)_{\Sigma} \wedge e^{\mathcal{F}} = \int_{\Sigma} \mathrm{d}\sigma\, \mathcal{E}(\Sigma, \mathcal{F}) \ . \tag{3.9}$$

For type II backgrounds, the generalized calibration form ω can be constructed explicitly as follows. As explained in [13] one has to break the $SO(1,9)$ symmetry of the tangent bundle of spacetime to $SO(9)$. We decompose the Killing spinors:

$$\epsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \chi_1 \, , \qquad \epsilon_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \chi_2 \quad (\text{IIB})\, , \quad \epsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \chi_2 \quad (\text{IIA})\, , \tag{3.10}$$

[9] Alternatively the calibration form is sometimes defined to obey:

$$\left(\omega' \wedge e^{\mathcal{F}}\right)_{\Sigma} \leq \mathrm{d}\sigma \mathcal{E}(\Sigma, \mathcal{F}) \ ,$$

as well as the differential condition:

$$\mathrm{d}_H \omega' = 0 \ .$$

The two definitions are related by: $\omega' = \omega - \delta_{q,d}C^{\text{el}}$; the one we adopt in the main text is more natural from the point of view of the calibrations/background supersymmetry correspondence.

where $\chi_{1,2}$ are real, commuting spinors of $SO(9)$. The ten-dimensional gamma matrices decompose accordingly as

$$\Gamma_0 = (i\sigma_2) \otimes \mathbb{1} \, , \qquad \Gamma_m = \sigma_1 \otimes \gamma_m \, , \qquad \Gamma_{11} = \sigma_3 \otimes \mathbb{1} \, , \tag{3.11}$$

where σ_i are the Pauli matrices, and $\gamma_m = e_m{}^a \gamma_a$ are nine-dimensional gamma matrices, with $e_m{}^a$ the (warped) vielbein associated with the metric in (3.1).

Using the $SO(9)$ spinors $\chi_{1,2}$ one can construct on the nine-dimensional space the real polyform

$$\Omega := \sum_{p \text{ even/odd}} \frac{e^{A-\Phi}}{p!|a|^2} \left(\widetilde{\chi}_1 \gamma_{m_1 \ldots m_p} \chi_2\right) \mathrm{d}x^{m_1} \wedge \ldots \wedge \mathrm{d}x^{m_p} \, , \tag{3.12}$$

where one has to sum over p even/odd in IIA/IIB respectively, and we have normalized:

$$\widetilde{\chi}_1 \chi_1 = \widetilde{\chi}_2 \chi_2 = |a|^2 \, . \tag{3.13}$$

The equality of the norms of $\chi_{1,2}$ can be seen to follow from the requirement that the background admits kappa-symmetric branes which do not break the supersymmetry of the background. In this case, it can be shown on rather general grounds that supersymmetry implies $|a|^2 \propto e^A$. We will choose the proportionality constant so that:

$$|a|^2 = e^A \, . \tag{3.14}$$

Let us denote by $\Omega^{(q-1)}$ the sum of all terms in (3.12) which contain exactly $(q-1)$ external spatial directions. Slightly adapting the proof in appendix A.3 of [14] to the present setup, it can then be seen that the following polyform:

$$\omega^{(d-q)} := \frac{\Omega^{(q-1)}}{\mathrm{vol}_{\mathrm{sp}}^{(q-1)}} \, , \tag{3.15}$$

is a calibration form for static, magnetic D-branes filling q external spacetime directions. In the above $\mathrm{vol}_{\mathrm{sp}}^{(q-1)}$ is the unwarped volume density along the $(q-1)$ external spatial directions that the brane fills, and the superscript of ω denotes the codimension with respect to the external d-dimensional spacetime. Note that this is *not* in general equal to the codimension of the branes with respect to the ten-dimensional spacetime: the branes wrap p-dimensional cycles in the internal space such that $p + q =$ odd/even in IIA/IIB.

Indeed it can be seen that $\omega^{(d-q)}$ defined in eq. (3.15) satisfies the algebraic inequality (3.7). Moreover, for $\omega^{(d-q)}$ to satisfy the differential condition (3.8), it suffices that [14]:

$$\iota_{v_+} F = 0 \; ; \quad \text{and} \quad v_- = 0 \, , \tag{3.16}$$

where the vectors $v_\pm$ are given by:

$$v_\pm^m := \begin{cases} (\widetilde{\chi}_1 \gamma^m \chi_1) \mp (\widetilde{\chi}_2 \gamma^m \chi_2) \, , & \text{in IIA} \\ (\widetilde{\chi}_1 \gamma^m \chi_1) \pm (\widetilde{\chi}_2 \gamma^m \chi_2) \, , & \text{in IIB} \end{cases} \, . \tag{3.17}$$

It can be easily verified that the conditions in eq. (3.16) are automatically satisfied for all the backgrounds which will be considered in sections 4.1 - 4.4.

4. Calibrations and supersymmetry in diverse dimensions

We will now apply the method described in section 3.1 to construct the calibration forms for all supersymmetric backgrounds of the type described at the beginning of section 3.

We start by reviewing the well-known $d = 4$ case in section 4.1. The supersymmetry equations can be cast in the form of four real first-order differential equations for two (complex) pure spinors of Cliff(6, 6), one of which imposes the GCY condition [4]. Moreover, the result of the calibration analysis is that there is a one-to-one correspondence between the supersymmetry pure-spinor equations for the background (assuming the background admits calibrations so that eq. (3.13) holds), and the differential equations obeyed by the generalized calibrations in that background [12].

We then repeat the analysis for the $d = 6$ case in section 4.2. We construct the generalized calibrations for the background and we express the differential equations which they obey as a set of five real first-order equations for two (complex) pure spinors of Cliff(4, 4). A brute force calculation given in appendix B shows that, as for the $d = 4$ case, the content of these five real pure-spinor equations is precisely equivalent to the supersymmetry equations for the background – assuming the background admits calibrations. Moreover, as in the $d = 4$ case, one of the consequences of supersymmetry is that the internal manifold is GCY.

The remaining two cases, $d = 8, 2$, are discussed in sections 4.3, 4.4 respectively. We work out the differential conditions for the generalized calibrations in these backgrounds and express them as first-order differential equations for two (complex) pure spinors. As in the previous two cases, the equations imply that the internal manifold is GCY. For these last two cases we do not verify that the differential conditions thus obtained are equivalent to the superymmetry equations – although we conjecture it to be true, based on the results of the $d = 4, 6$ cases. The result of the analysis for all d is summarized in eq. (1.1) and table 1 of the introduction.

Before we proceed to the case-by-case analysis, let us also mention that the spinor ansatz of eq. (3.5) must be modified in order to take into account the $SO(1, 9) \rightarrow SO(9)$ reduction of eq. (3.10). Under $SO(1, d-1) \rightarrow SO(d-1)$ the Weyl spinor ζ, which transforms in the $2_+^{d/2-1}$ of $SO(1, d-1)$, restricts to the $2^{d/2-1}$ of $SO(d-1)$; we will denote the restriction to $SO(d-1)$ by θ. The spinor ansatz then takes the form of eq. (3.10), with:

$$\chi_i = \frac{1}{\sqrt{2}} \left(\theta \otimes \eta_i + \text{c.c.}\right) \ , \tag{4.1}$$

where $i = 1, 2$ and η_i are the same pure Weyl spinors of Cliff$(10-d)$ as in eq. (3.5). We assume the normalization: $\widetilde{\theta}\theta^c = 1$, $\widetilde{\eta}_i\eta_i^c = |a|^2$, $i = 1, 2$, so that (3.13) is obeyed. The precise form of the complex conjugate on the right-hand sides of the equations above will be given explicitly for each case in the following.

This restriction on the norms of the η_i's (following from the requirement that the background admit calibrated D-branes) implies, taking eq. (3.14) into account, that the pair of compatible pure spinors defined in eq. (2.2) are non-vanishing provided the warp factor

A is finite. We will assume that they are also globally defined; as reviewed in section 2, this implies the reduction of the structure group[10] of the direct sum of the tangent and cotangent bundles of the internal manifold to $SU(k) \times SU(k)$, where $k := (10-d)/2$.

4.1 d=4

The pure spinor equations for supersymmetric backgrounds of the form $\mathbb{R}^{1,3} \times \mathcal{M}_6$ with two complex supercharges, *i.e.* minimal supersymmetry in four dimensions, were worked out in [4]. In [12] it was subsequently shown that the differential conditions for generalized calibrations for static, magnetic D-branes in this background are in one-to-one correspondence with the supersymmetry equations in pure-spinor form. We shall review this case here for completeness.

Under $SO(9) \to SO(3) \times SO(6)$ the nine-dimensional gamma matrices and charge conjugation matrix decompose as:

$$\Gamma_i = \sigma_i \otimes \gamma_7 \, , \qquad \Gamma_{m+3} = \mathbb{1} \otimes \gamma_m \, , \qquad C_9 = C_3 \otimes \gamma_7 C_6 \ , \tag{4.2}$$

where $\{\sigma_i, \ i = 1, 2, 3\}$, $\{\gamma_m, \ m = 1, \ldots, 6\}$ are three-, six-dimensional gamma matrices, respectively, and γ_7 is the six-dimensional chirality matrix. In our spinor conventions, described in appendix A, it can then be seen that the explicit form of the spinor decomposition eq. (4.1) reads:

$$\chi_1 = \frac{1}{\sqrt{2}} \left(\theta \otimes \eta_1 - \theta^c \otimes \eta_1^c\right) \ ; \qquad \chi_2 = \frac{1}{\sqrt{2}} \left(\theta \otimes \eta_2 \pm \theta^c \otimes \eta_2^c\right) \ , \tag{4.3}$$

where θ is in the $\mathbf{2}$ of $SO(3)$ and η_1, η_2 are Weyl spinors in the $\mathbf{4}$ of $SO(6)$, with $\gamma_7\eta_1 = \eta_1$ and $\gamma_7\eta_2 = \mp\eta_2$ in IIA/IIB.

Plugging the above expressions for $\chi_{1,2}$ into eq. (3.12, 3.15), taking eqs. (A.10, A.11, A.13) into account, we find that the only non-vanishing calibration forms occur at codimensions zero (spacetime-filling), one (domain walls), and two (D-strings) with respect to the external spacetime. The explicit expressions read:

$$\begin{aligned} \omega^{(0)} &= e^{4A-\Phi}\mathrm{Re}\Psi_1 \\ \omega^{(1)} &= e^{3A-\Phi}\mathrm{Im}\left(e^{i\varphi}\Psi_2\right) \\ \omega^{(2)} &= e^{2A-\Phi}\mathrm{Im}\Psi_1 \ , \end{aligned} \tag{4.4}$$

where we have taken eq. (2.2) into account. The phase φ on the right hand side of the second line comes from the normalization:

$$\frac{1}{2}(\widetilde{\theta}\sigma_{ij}\theta)\,\mathrm{d}x^i \wedge \mathrm{d}x^j = e^{i\varphi}\mathrm{vol}_{\mathrm{sp}}^{(2)} \ , \tag{4.5}$$

where as in eq. (3.15) $\mathrm{vol}_{\mathrm{sp}}^{(q-1)}$ is the unwarped volume density along the $(q-1)$-dimensional external space that the brane fills.

[10] The reduction of the structure group of the internal manifold is, in general, *not* a necessary condition for supersymmetric backgrounds [17]; see [18] for a recent explicit example.

The connection between the generalized calibrations (4.4) and background supersymmetry is made by taking the differential equation (3.8) into account. We thus obtain the following equations:

$$\begin{aligned} \mathrm{d}_H \left(e^{4A-\Phi}\mathrm{Re}\Psi_1\right) &= F^{\mathrm{el}} \\ \mathrm{d}_H \left(e^{2A-\Phi}\mathrm{Im}\Psi_1\right) &= 0 \\ \mathrm{d}_H \left(e^{3A-\Phi}\Psi_2\right) &= 0 \ , \end{aligned} \tag{4.6}$$

which are equivalent to the background supersymmetry equations [4]. Note that the last equation above is precisely the GCY condition for the pure spinor $e^{3A-\Phi}\Psi_2$; it is obtained by imposing $\mathrm{d}_H\omega^{(1)} = 0$ for all φ, with $\omega^{(1)}$ given in eq. (4.4).

The background supersymmetry equations (4.6) and their correspondence with the D-brane calibrations eq. (4.4) is summarized in the $d = 4$ column of table 1 and eq. (1.1) of the introduction.

4.2 d=6

Let us now consider supersymmetric backgrounds of the form $\mathbb{R}^{1,5} \times \mathcal{M}_4$ preserving four complex supercharges, *i.e.* minimal supersymmetry in six dimensions. We will show that, as in the $d = 4$ case, the differential conditions for generalized calibrations for static, magnetic D-branes in this background are in one-to-one correspondence with the supersymmetry equations in pure-spinor form.

Under $SO(9) \to SO(5) \times SO(4)$ the nine-dimensional gamma matrices and charge conjugation matrix decompose as:

$$\Gamma_i = \sigma_i \otimes \gamma_5 \, , \qquad \Gamma_{m+5} = \mathbb{1} \otimes \gamma_m \, , \qquad C_9 = C_5 \otimes C_4 \ , \tag{4.7}$$

where $\{\sigma_i, \ i = 1, \ldots, 5\}$, $\{\gamma_m, \ m = 1, \ldots, 4\}$ are five-, four-dimensional gamma matrices, respectively, and γ_5 is the four-dimensional chirality matrix. It can then be seen that the explicit form of the spinor decomposition eq. (4.1) reads:

$$\chi_i = \frac{1}{\sqrt{2}} \left(\theta \otimes \eta_i + \theta^c \otimes \eta_i^c\right) \ , \tag{4.8}$$

where $i = 1, 2$; θ is in the **4** of $SO(5)$ and η_1, η_2 are Weyl spinors in the **2** of $SO(4)$, with $\gamma_5\eta_1 = \eta_1$ and $\gamma_7\eta_2 = \mp\eta_2$ in IIA/IIB.

Plugging the above expressions for $\chi_{1,2}$ into eq. (3.12, 3.15), taking eqs. (A.10, A.11, A.13) into account, we find that the only non-vanishing calibration forms occur at codimensions zero (spacetime-filling), two and four with respect to the external spacetime. The explicit expressions read:

$$\begin{aligned} \omega^{(0)} &= e^{6A-\Phi}\mathrm{Re}\Psi_1 \\ \omega^{(2)} &= e^{4A-\Phi}\mathrm{Re}\left(e^{i\varphi}\Psi_2\right) + e^{4A-\Phi}\mathrm{Im}\Psi_1 \\ \omega^{(4)} &= e^{2A-\Phi}\mathrm{Re}\Psi_1 \ , \end{aligned} \tag{4.9}$$

where we have taken eq. (2.2) into account. The phase φ on the right hand side of the second line comes from the normalization:

$$\frac{1}{3!}(\widetilde{\theta}\sigma_{ijk}\theta)\,\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k = e^{i\varphi}\mathrm{vol}^{(3)}_{\mathrm{sp}}\ , \tag{4.10}$$

where as in eq. (3.15) $\mathrm{vol}^{(q-1)}_{\mathrm{sp}}$ is the unwarped volume density along the $(q-1)$-dimensional external space that the brane fills.

The connection between the generalized calibrations (4.9) and background supersymmetry is made by taking the differential equation (3.8) into account. We thus obtain the following equations:

$$\begin{aligned} \mathrm{d}_H\left(e^{6A-\Phi}\mathrm{Re}\Psi_1\right) &= F^{\mathrm{el}}\\ \mathrm{d}_H\left(e^{4A-\Phi}\mathrm{Im}\Psi_1\right) &= 0\\ \mathrm{d}_H\left(e^{2A-\Phi}\mathrm{Re}\Psi_1\right) &= 0\\ \mathrm{d}_H\left(e^{4A-\Phi}\Psi_2\right) &= 0\ . \end{aligned} \tag{4.11}$$

The last equation above is precisely the GCY condition for the pure spinor $e^{4A-\Phi}\Psi_2$; that and the equation in the second line above are obtained by imposing $\mathrm{d}_H\omega^{(2)}=0$ for all φ, with $\omega^{(2)}$ given in eq. (4.9).

A tedious but straightforward brute-force calculation, given in appendix B, shows that the content of eqs. (4.11) is precisely equivalent to the supersymmetry equations for the background, thus proving the one-to-one correspondence between supersymmetry and D-brane calibrations. This correspondence is summarized in the $d=6$ column of table 1 and eq. (1.1) of the introduction.

4.3 d=8

Let us consider supersymmetric backgrounds of the form $\mathbb{R}^{1,7}\times\mathcal{M}_2$ preserving eight complex supercharges, *i.e.* minimal supersymmetry in eight dimensions.

Under $SO(9)\to SO(7)\times SO(2)$ the nine-dimensional gamma matrices and charge conjugation matrix decompose as:

$$\Gamma_i = \sigma_i\otimes\gamma_3\,,\qquad \Gamma_{m+7} = \mathbb{1}\otimes\gamma_m\,,\qquad C_9 = C_7\otimes\gamma_3 C_2\ , \tag{4.12}$$

where $\{\sigma_i,\ i=1,\ldots,7\}$, $\{\gamma_m,\ m=1,2\}$ are seven-, two-dimensional gamma matrices, respectively, and γ_3 is the two-dimensional chirality matrix. It can then be seen that the explicit form of the spinor decomposition eq. (4.1) reads:

$$\chi_1 = \frac{1}{\sqrt{2}}\left(\theta\otimes\eta_1 - \theta^c\otimes\eta_1^c\right)\ ;\quad \chi_2 = \frac{1}{\sqrt{2}}\left(\theta\otimes\eta_2 \pm \theta^c\otimes\eta_2^c\right)\ , \tag{4.13}$$

where θ is in the **8** of $SO(7)$ and η_1, η_2 are Weyl spinors in the **1** of $SO(2)$, with $\gamma_3\eta_1=\eta_1$ and $\gamma_3\eta_2=\mp\eta_2$ in IIA/IIB.

Plugging the above expressions for $\chi_{1,2}$ into eq. (3.12, 3.15), taking eqs. (A.10, A.11, A.13) into account, we find that the only non-vanishing calibration forms occur at codimensions zero (spacetime-filling), two, three, four, six and seven with respect to the external spacetime. The explicit expressions read:

$$\begin{aligned}
\omega^{(0)} &= e^{8A-\Phi}\mathrm{Re}\Psi_1 \\
\omega^{(2)} &= e^{6A-\Phi}\mathrm{Im}\Psi_1 \\
\omega^{(3)} &= e^{5A-\Phi}\mathrm{Im}(e^{i\varphi}\Psi_2) \\
\omega^{(4)} &= e^{4A-\Phi}\mathrm{Re}\Psi_1 \\
\omega^{(6)} &= e^{2A-\Phi}\mathrm{Im}\Psi_1 \\
\omega^{(7)} &= e^{A-\Phi}\mathrm{Im}(e^{i\xi}\Psi_2) \ ,
\end{aligned} \tag{4.14}$$

where we have taken eq. (2.2) into account and we have set: $(\widetilde{\theta}\theta) = e^{i\xi}$. Moreover, the phase φ on the right hand side of the third line comes from the normalization:

$$\frac{1}{4!}(\widetilde{\theta}\sigma_{i_1\dots i_4}\theta)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_4} = e^{i\varphi}\mathrm{vol}^{(4)}_{\mathrm{sp}} \ , \tag{4.15}$$

where as in eq. (3.15) $\mathrm{vol}^{(q-1)}_{\mathrm{sp}}$ is the unwarped volume density along the $(q-1)$-dimensional external space that the brane fills.

Taking into account the fact that the generalized calibrations (4.14) obey the differential equation (3.8), we obtain the following equations:

$$\begin{aligned}
\mathrm{d}_H\left(e^{8A-\Phi}\mathrm{Re}\Psi_1\right) &= F^{\mathrm{el}} \\
\mathrm{d}_H\left(e^{6A-\Phi}\mathrm{Im}\Psi_1\right) &= 0 \\
\mathrm{d}_H\left(e^{4A-\Phi}\mathrm{Re}\Psi_1\right) &= 0 \\
\mathrm{d}_H\left(e^{2A-\Phi}\mathrm{Im}\Psi_1\right) &= 0 \\
\mathrm{d}_H\left(e^{5A-\Phi}\Psi_2\right) &= 0 \\
\mathrm{d}_H\left(e^{A-\Phi}\Psi_2\right) &= 0 \ .
\end{aligned} \tag{4.16}$$

Note that the last two equations above are precisely the GCY condition for the pure spinors $e^{5A-\Phi}\Psi_2$, $e^{A-\Phi}\Psi_2$, respectively; they are obtained by imposing $\mathrm{d}_H\omega^{(3,7)} = 0$ for all φ, ξ; with $\omega^{(3,7)}$ given in eq. (4.14).

Based on the results for $d = 4, 6$, we conjecture that the content of eqs. (4.16) should be precisely equivalent to the supersymmetry equations for the background. Assuming the correspondence to be true, the results of this section are summarized in the $d = 8$ column of table 1 and eq. (1.1) of the introduction.

4.4 d=2

Let us consider supersymmetric backgrounds of the form $\mathbb{R}^{1,1}\times\mathcal{M}_8$ preserving one complex supercharge, corresponding to $\mathcal{N} = (2, 0)$ supersymmetry in two dimensions.

Under $SO(9) \to SO(8)$ the nine-dimensional gamma matrices and charge conjugation matrix decompose as:

$$\Gamma_m = \gamma_m \,, \qquad \Gamma_9 = \gamma_9 \,, \qquad C_9 = C_8 \,, \tag{4.17}$$

where $\{\gamma_m,\ m = 1, \ldots, 8\}$ are eight-dimensional gamma matrices, and γ_9 is the eight-dimensional chirality matrix. It can then be seen that the explicit form of the Killing spinor decomposition eq. (4.1) reads:

$$\chi_i = \frac{1}{\sqrt{2}} \left(e^{\frac{i\varphi}{2}} \eta_i + e^{-\frac{i\varphi}{2}} \eta_i^c \right) \,, \tag{4.18}$$

where $i = 1, 2$, φ is a phase and η_1, η_2 are pure Weyl spinors[11] in the **8** of $SO(8)$, with $\gamma_9 \eta_1 = \eta_1$ and $\gamma_9 \eta_2 = \mp \eta_2$ in IIA/IIB.

Plugging the above expressions for $\chi_{1,2}$ into eq. (3.12, 3.15), taking eqs. (A.10, A.11, A.13) into account, we find that the only non-vanishing calibration form occurs at codimensions zero (spacetime-filling). The explicit expression reads:

$$\omega^{(0)} = e^{2A-\Phi} \mathrm{Re} \left(e^{i\varphi} \Psi_2 \right) + e^{2A-\Phi} \mathrm{Re} \Psi_1 \,, \tag{4.20}$$

where we have taken eq. (2.2) into account. The phase φ on the right hand side is the same as in eq. (4.18).

Taking into account the fact that the generalized calibrations (4.20) obey the differential equation (3.8), we obtain the following equations:

$$\begin{aligned} \mathrm{d}_H \left(e^{2A-\Phi} \mathrm{Re} \Psi_1 \right) &= F^{\mathrm{el}} \\ \mathrm{d}_H \left(e^{2A-\Phi} \Psi_2 \right) &= 0 \,. \end{aligned} \tag{4.21}$$

The equations above are obtained by imposing $\mathrm{d}_H \omega^{(0)} = F^{\mathrm{el}}$ for all φ, with $\omega^{(0)}$ given in eq. (4.20). The second of the two equations is precisely the GCY condition for the pure spinor $e^{2A-\Phi} \Psi_2$.

[11] Note that, as already mentioned, not all Weyl spinors of $SO(8)$ are pure. If η is a pure Weyl spinor, the purity of η can be seen to be equivalent to the condition:

$$\widetilde{\eta} \eta = 0 \,. \tag{4.19}$$

This condition can only be satisfied (for non-vanishing η) if η is complexified. Given a pair η_R, η_I of orthogonal Majorana-Weyl spinors of $SO(8)$ of the same chirality:

$$\widetilde{\eta}_R \eta_R = \widetilde{\eta}_I \eta_I = |a|^2 \,; \qquad \widetilde{\eta}_R \eta_I = 0 \,,$$

one can construct the complexified pure spinor η through:

$$\eta := \frac{1}{\sqrt{2}} (\eta_R + i \eta_I) \,.$$

These are precisely the conditions imposed on the internal part of the Killing spinor in $\mathcal{N} = 2$ M-theory compactifications on eight-manifolds of the type considered in [19].

Based on the results for $d = 4, 6$, we conjecture that the content of eqs. (4.21) should be precisely equivalent to the supersymmetry equations for the background. Assuming the correspondence to be true, the results of this section are summarized in the $d = 2$ column of table 1 and eq. (1.1) of the introduction.

5. Conclusions

We considered type II backgrounds of the form $\mathbb{R}^{1,d-1} \times \mathcal{M}_{10-d}$ for even d, preserving $2^{d/2-1}$ complex supercharges – as many as the components of a complexified Weyl spinor of $SO(1, d-1)$. For $d = 6$ we proved that there is a one-to-one correspondence between background supersymmetry equations in pure-spinor form and D-brane generalized calibrations – a fact which was already known in the $d = 4$ case. We conjectured that this one-to-one correspondence should hold for general d, and used this to 'predict' the background supersymmetry equations in pure-spinor form for the $d = 2, 8$ cases. It would be nice to verify our conjecture for $d = 2, 8$ by either a brute-force computation, as we have done here in the $d = 6$ case, or by a counting argument, as in [20] for the $d = 4$ case.

We expect our results for the background supersymmetry equations in pure-spinor form to be useful in finding novel flux vacua. Of course in this case one would also have to solve the Bianchi identities in addition to the supersymmetry equations [21, 22, 13]. The study of generalized calibrations has shed light to the construction of effective actions, and has recently suggested a way to break supersymmetry in a controlled way [16, 23]. It would be interesting to pursue this connection further. It would also be interesting to repeat our analysis for backgrounds of the form $\mathrm{AdS}_d \times \mathcal{M}_{10-d}$, along the lines of [14].

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A. Spinors and gamma matrices in Euclidean spaces

In this section we list some useful relations and explain in more detail our spinor conventions for general even-dimensional Euclidean spaces of dimension $2k$.

The charge conjugation matrix obeys:

$$C^{\mathrm{Tr}} = (-)^{\frac{1}{2}k(k+1)}C \ ; \quad C^* = (-)^{\frac{1}{2}k(k+1)}C^{-1} \ ; \quad \gamma_m^{\mathrm{Tr}} = (-)^k C^{-1}\gamma_m C \ . \tag{A.1}$$

The complex conjugate η^c of a spinor η is given by:

$$\eta^c := C\eta^* \ , \tag{A.2}$$

form which it follows that:

$$(\eta^c)^c = (-)^{\frac{1}{2}k(k+1)}\eta \ . \tag{A.3}$$

The chirality matrix γ_{2k+1} is defined by:

$$\gamma_{2k+1} := i^k \gamma_1 \dots \gamma_{2k} \ , \tag{A.4}$$

and obeys

$$\gamma_{2k+1}^{\rm Tr} = (-)^k C^{-1} \gamma_{2k+1} C \ , \tag{A.5}$$

as follows from eqs. (A.4, A.1). The chirality projector:

$$P_\pm := \frac{1}{2}(1 \pm \gamma_{2k+1}) \ , \tag{A.6}$$

projects a Dirac spinor χ onto the chiral, antichiral Weyl parts $\chi_\pm$:

$$\chi_\pm = P_\pm \chi \ . \tag{A.7}$$

Taking eq. (A.5) into account we obtain:

$$C^{-1} P_\pm = \begin{cases} P_\pm^{\rm Tr} C^{-1} \ , & k = \text{even} \\ P_\mp^{\rm Tr} C^{-1} \ , & k = \text{odd} \end{cases} \ . \tag{A.8}$$

Covariantly-transforming spinor bilinears must be of the form $(\widetilde{\psi} \gamma_{m_1 \dots m_p} \chi)$, where in any dimension we define:

$$\widetilde{\psi} := \psi^{\rm Tr} C^{-1} \ . \tag{A.9}$$

Using eq. (A.8) we find:

$$\begin{aligned} (\widetilde{\psi}_\pm \gamma_{m_1 \dots m_{2l}} \chi_\mp) = 0 = (\widetilde{\psi}_\pm \gamma_{m_1 \dots m_{2l+1}} \chi_\pm) \ , \quad k = \text{even} \\ (\widetilde{\psi}_\pm \gamma_{m_1 \dots m_{2l}} \chi_\pm) = 0 = (\widetilde{\psi}_\pm \gamma_{m_1 \dots m_{2l+1}} \chi_\mp) \ , \quad k = \text{odd} \ . \end{aligned} \tag{A.10}$$

Moreover:

$$(\widetilde{\psi} \gamma_{m_1 \dots m_p} \chi) = (-)^{kp + \frac{1}{2}k(k+1)} (\widetilde{\chi} \gamma_{m_p \dots m_1} \psi) = (-)^{\frac{1}{2}(k-p)(k-p+1)} (\widetilde{\chi} \gamma_{m_1 \dots m_p} \psi) \ . \tag{A.11}$$

The identity

$$\gamma^*_{m_1 \dots m_p} = (-)^{kp} C^{-1} \gamma_{m_1 \dots m_p} C \ , \tag{A.12}$$

can be used to show the following relations:

$$\begin{aligned} (\widetilde{\psi} \gamma_{m_1 \dots m_p} \chi)^* = (-)^{kp} (\widetilde{\psi^c} \gamma_{m_1 \dots m_p} \chi^c) \\ (\widetilde{\psi} \gamma_{m_1 \dots m_p} \chi^c)^* = (-)^{kp + \frac{1}{2}k(k+1)} (\widetilde{\psi^c} \gamma_{m_1 \dots m_p} \chi) \ . \end{aligned} \tag{A.13}$$

B. Explicit solution of the supersymmetry equations in $d = 6$

In this section we give the details of the derivation of the explicit solution of the Killing spinor equations for type II $\mathbb{R}^{1,5} \times \mathcal{M}_4$ flux backgrounds with minimal supersymmetry in six dimensions. The IIA, IIB cases are treated separately in sections B.1, B.2 below. The requirement that the background admits a pair of globally-defined, nowhere-vanishing

pure spinors, leads to a different topological condition in each case: on the type IIA side it implies the trivialization of the structure group of $T\mathcal{M}_4$, while on the IIB side it implies the reduction of the structure group of $T\mathcal{M}_4$ to $SU(2)$; see [24, 25] for a recent discussion.

The explicit solution of the Killing spinor equations (*i.e.* the supersymmetry conditions) given below can be seen to be identical to the solution of the set of pure-spinor equations (4.11) of section 4.2 – which are the differential conditions obeyed by static, magnetic D-brane calibrations. Thus we provide here for the $d = 6$ case a proof of the one-to-one correspondence between background supersymmetry pure-spinor equations and D-brane calibrations.

Our starting point is the ten-dimensional supersymmetry equations:

$$\begin{aligned}
0 &= \left(\partial\!\!\!/\Phi + \frac{1}{2}H\!\!\!\!/\right)\epsilon_1 + \left(\frac{1}{16}e^{\Phi}\Gamma^M F\!\!\!\!/\,\Gamma_M\Gamma_{11}\right)\epsilon_2 \\
0 &= \left(\partial\!\!\!/\Phi - \frac{1}{2}H\!\!\!\!/\right)\epsilon_1 - \left(\frac{1}{16}e^{\Phi}\Gamma^M \sigma(F\!\!\!\!/)\Gamma_M\Gamma_{11}\right)\epsilon_1 \\
0 &= \left(\nabla_M + \frac{1}{4}H\!\!\!\!/_M\right)\epsilon_1 + \left(\frac{1}{16}e^{\Phi}F\!\!\!\!/\,\Gamma_M\Gamma_{11}\right)\epsilon_2 \\
0 &= \left(\nabla_M - \frac{1}{4}H\!\!\!\!/_M\right)\epsilon_2 - \left(\frac{1}{16}e^{\Phi}\sigma(F\!\!\!\!/)\Gamma_M\Gamma_{11}\right)\epsilon_1 \ ,
\end{aligned} \tag{B.1}$$

where the ten-dimensional Majorana-Weyl Killing spinors are decomposed as in (3.5). The ten-dimensional gamma matrices Γ_M are decomposed as follows:

$$\Gamma_\mu = \hat{\gamma}_\mu \otimes \mathbb{1}\,, \qquad \Gamma_{m+5} = \hat{\gamma}_7 \otimes \gamma_m \ , \tag{B.2}$$

where $\{\hat{\gamma}_\mu,\ \mu = 0,\ldots,5\}$, $\{\gamma_m,\ m = 1,\ldots,4\}$ are six-, four-dimensional gamma matrices, respectively, and $\hat{\gamma}_7$ is the six-dimensional chirality matrix. In the following subsections we will consider the IIA, IIB cases separately.

B.1 IIA

We may parameterize the internal nowhere-vanishing, globally-defined Weyl spinors $\eta_{1,2}$ in the Killing-spinor ansatz in (3.5) as follows:

$$\eta_1 = a\,\eta \ , \quad \eta_2 = b\,\chi \ , \tag{B.3}$$

where η, χ are unimodular Weyl spinors of opposite chirality and $|a|^2 = |b|^2$. Moreover we can choose without loss of generality the phases of η, χ so that $a = b \in \mathbb{R}$.

The pair of nowhere-vanishing, globally-defined Weyl spinors η, χ trivializes the tangent bundle of $\mathcal{M}_4$, so that the structure group reduces to $\mathbb{1}$. This can also be seen by constructing a pair of complex vectors:

$$u^m = \widetilde{\eta}\gamma^m\chi \ ; \quad v^m = \widetilde{\eta}\gamma^m\chi^c \ . \tag{B.4}$$

As can be proven by Fierzing, the four real globally-defined vectors Reu, Imu, Rev, Imv are unimodular and mutually orthogonal; hence they provide an explicit trivialization of the tangent bundle $T\mathcal{M}_4$.

Let us also mention that in deriving the general solution to the Killing spinor equations, it will be useful to take the following relations into account:

$$\begin{aligned} \gamma_m\eta &= v_m\chi - u_m\chi^c \\ \gamma_m\eta^c &= v^*_m\chi^c + u^*_m\chi \\ \gamma_m\chi &= v^*_m\eta + u_m\eta^c \\ \gamma_m\chi^c &= v_m\eta^c - u^*_m\eta \ , \end{aligned} \tag{B.5}$$

which can be shown by Fierzing.

We now proceed by decomposing all forms on the basis of u, v – which can also be thought of as one-forms given the existence of a metric on $\mathcal{M}_4$; in the following we will use the same notation for both the vectors and the one forms.

The most general decomposition of the various components of the (magnetic) RR flux F, *cf.* eq. (3.3), reads as follows:

$$F = F_0 + F_2 + F_4 \ , \tag{B.6}$$

with:

$$\begin{aligned} e^\Phi F_0 &= f^{(0)} \\ e^\Phi F_2 &= \frac{1}{2}(\mathrm{i}f^{(2)}_1 u\wedge u^* + \mathrm{i}f^{(2)}_2 v\wedge v^* + f^{(2)}_3 u\wedge v + f^{(2)*}_3 u^*\wedge v^* + f^{(2)}_4 u\wedge v^* + f^{(2)*}_4 u^*\wedge v) \\ e^\Phi F_4 &= \frac{1}{4} f^{(4)} u\wedge v\wedge u^*\wedge v^* \ , \end{aligned} \tag{B.7}$$

where $f^{(0)}$, $f^{(2)}_{1,2}$, $f^{(4)}$ are real scalars, and $f^{(2)}_{3,4}$ are complex scalars. Similarly, we decompose the NSNS three-form as follows:

$$H = \star_4 \left(h_1 u + h_2 v + \text{c.c.}\right) \ , \tag{B.8}$$

where $h_{1,2}$ are complex scalars.

We also need the decompositions of the derivatives of the real scalars Φ, A, a:

$$\begin{aligned} \partial_m\Phi &= \tfrac{1}{2}\left(u^*_m\varphi_u + v^*_m\varphi_v + \text{c.c.}\right) \\ \partial_m A &= \tfrac{1}{2}\left(u^*_m A_u + v^*_m A_v + \text{c.c.}\right) \\ \partial_m a &= \tfrac{1}{2}\left(u^*_m(\partial a)_u + v^*_m(\partial a)_v + \text{c.c.}\right) \ , \end{aligned} \tag{B.9}$$

where φ_u, φ_v, A_u, A_v, $(\partial a)_u$, $(\partial a)_v$, are complex scalars.

The torsion classes of the (trivial) structure of $T\mathcal{M}_4$ parameterize the failure of η, χ to be covariantly constant. Explicitly, we define the torsion classes $\mathcal{W}^{(i)}_m$, $i = 1, \ldots 4$, via:

$$\begin{aligned} \nabla_m\eta &= \mathcal{W}^{(1)}_m\eta + \mathcal{W}^{(2)}_m\eta^c \\ \nabla_m\chi &= \mathcal{W}^{(3)}_m\chi + \mathcal{W}^{(4)}_m\chi^c \ , \end{aligned} \tag{B.10}$$

where $\mathcal{W}^{(2,4)}$ are complex one-forms, and $\mathcal{W}^{(1,3)}$ are imaginary one-forms; the latter property follows from the definition (B.10) upon taking the unimodularity of η, χ into account. Explicitly, for $i = 1, \ldots, 4$ we decompose:

$$\mathcal{W}^{(i)} = \tfrac{1}{2}(u^*\mathcal{W}_u^{(i)} + v^*\mathcal{W}_v^{(i)} + u\mathcal{W}_{u^*}^{(i)} + v\mathcal{W}_{v^*}^{(i)}) \ , \tag{B.11}$$

where $\mathcal{W}_u^{(i)}$ $\mathcal{W}_v^{(i)}$, $\mathcal{W}_{u^*}^{(i)}$, $\mathcal{W}_{v^*}^{(i)}$ are complex scalars. Moreover, the fact that $\mathcal{W}^{(1,3)}$ are imaginary implies:

$$\mathcal{W}_{u^*}^{(i)} = -\mathcal{W}_u^{(i)} \ ; \quad \mathcal{W}_{v^*}^{(i)} = -\mathcal{W}_v^{(i)} \ , \tag{B.12}$$

for $i = 1, 3$. Let us also note that alternatively the torsion classes can be defined in terms of the exterior derivatives of u, v. Indeed, from eq. (B.10) we have, upon taking definition (B.4) into account:

$$\begin{aligned} \mathrm{d}u &= (\mathcal{W}^{(1)} + \mathcal{W}^{(3)}) \wedge u + \mathcal{W}^{(4)} \wedge v - \mathcal{W}^{(2)} \wedge v^* \\ \mathrm{d}v &= (\mathcal{W}^{(1)} - \mathcal{W}^{(3)}) \wedge v - \mathcal{W}^{(4)*} \wedge u + \mathcal{W}^{(2)} \wedge u^* \ . \end{aligned} \tag{B.13}$$

We are now ready to give the general solution to the background supersymmetry equations, by plugging the above expansions into the Killing spinor equations (B.1), taking eq. (B.5) into account. The solution is parameterized in terms of eight real unconstrained scalar degrees of freedom which we may take to be $f^{(0)}$, $f_1^{(2)}$, $f_4^{(2)}$, φ_u, φ_v (recall that the last two complex scalars parameterize $\partial_m\Phi$). Explicitly, the torsion classes are given by:

$$\begin{aligned} W_u^{(1)} &= \frac{1}{2}\varphi_u - f_4^{(2)*} \\ W_v^{(1)} &= \frac{1}{2}\varphi_v - \frac{\mathrm{i}}{4}f_1^{(2)} - \frac{1}{2}f^{(0)} \\ \\ W_u^{(2)} &= 0 \\ W_v^{(2)} &= -\frac{1}{2}f_4^{(2)*} \\ W_{u^*}^{(2)} &= \varphi_v - \mathrm{i}f_1^{(2)} - f^{(0)} \\ W_{v^*}^{(2)} &= -\varphi_u + \frac{3}{2}f_4^{(2)*} \\ \\ W_u^{(3)} &= \frac{1}{2}\varphi_u - f_4^{(2)*} \\ W_v^{(3)} &= -\frac{1}{2}\varphi_v + \frac{\mathrm{i}}{4}f_1^{(2)} + \frac{1}{2}f^{(0)} \\ \\ W_u^{(4)} &= 0 \\ W_v^{(4)} &= \varphi_u - \frac{3}{2}f_4^{(2)*} \\ W_{u^*}^{(4)} &= -\varphi_v^* - \mathrm{i}f_1^{(2)} + f^{(0)} \\ W_{v^*}^{(4)} &= \frac{1}{2}f_4^{(2)*} \ . \end{aligned} \tag{B.14}$$

The NSNS and RR fluxes are given by:

$$\begin{aligned}
h_1 &= -\varphi_u^* + \frac{3}{2} f_4^{(2)} \\
h_2 &= -\varphi_v^* - \frac{3\mathrm{i}}{4} f_1^{(2)} + \frac{5}{4} f^{(0)} \\
\\
f_2^{(2)} &= 0 \\
f_3^{(2)} &= -f_4^{(2)} \\
\\
f^{(4)} &= 0 \ .
\end{aligned} \tag{B.15}$$

Finally, the derivatives of a, A, *cf.* eq. (B.9), read:

$$\begin{aligned}
(\partial a)_u &= \frac{a}{4} f_4^{(2)*} \\
(\partial a)_v &= \frac{\mathrm{i}a}{8} f_1^{(2)} + \frac{a}{8} f^{(0)}
\end{aligned} \tag{B.16}$$

and

$$\begin{aligned}
A_u &= \frac{1}{2} f_4^{(2)*} \\
A_v &= \frac{\mathrm{i}}{4} f_1^{(2)} + \frac{1}{4} f^{(0)} \ .
\end{aligned} \tag{B.17}$$

We therefore see explicitly that $|a|^2 \propto e^A$, as already mentioned above eq. (3.14).

To make contact with the supersymmetry equations in pure-spinor form (4.11), we note that the definition (2.2) implies:

$$\begin{aligned}
\Psi_1 &= v - \frac{1}{2} u \wedge v \wedge u^* \\
\Psi_2 &= u + \frac{1}{2} u \wedge v \wedge v^* \ ,
\end{aligned} \tag{B.18}$$

where we have taken eqs. (B.3), (B.4), (B.5) into account. It is then straightforward to show that the solution of the Killing spinor equations given above is identical to the solution one obtains by substituting (B.18) into eqs. (4.11), taking (B.7) - (B.9), (B.13) into account.

B.2 IIB

We parameterize the internal nowhere-vanishing, globally-defined Weyl spinors $\eta_{1,2}$ in the Killing-spinor ansatz in (3.5) as follows:

$$\eta_1 = a\,\eta \ , \quad \eta_2 = b\,\eta + c\,\eta^c \ , \tag{B.19}$$

where η is a unimodular Weyl spinor of positive chirality. Without loss of generality we may choose the phase of η so that $a \in \mathbb{R}$; the scalars b, c are in general complex.

The nowhere-vanishing, globally-defined Weyl spinor η reduces the structure group of the tangent bundle of $\mathcal{M}_4$ to $SU(2)$. This can also be seen by constructing a real two-form j and a complex two-form ω on $\mathcal{M}_4$ as spinor bilinears:

$$j_{mn} = \mathrm{i}\tilde{\eta}\gamma_{mn}\eta^c \qquad\qquad \omega_{mn} = -\mathrm{i}\tilde{\eta}\gamma_{mn}\eta \ . \tag{B.20}$$

The pair (j, ω) defined above, can be seen by Fierzing to obey the definition of an $SU(2)$ structure:

$$j \wedge \omega = 0 \ ; \quad j \wedge j = \frac{1}{2}\omega \wedge \omega^* \neq 0 \ . \tag{B.21}$$

On $\mathcal{M}_4$ there is an almost complex structure, which can be given explicitly in terms of the projectors:

$$\left(\Pi^{\pm}\right)_m{}^n := \frac{1}{2}\left(\delta_m{}^n \mp i j_m{}^n\right) \ . \tag{B.22}$$

A one-form V can thus be decomposed into (1,0) and (0,1) parts V^+, V^- with respect to the almost complex structure via: $V^{\pm}_m := (\Pi^{\pm})_m{}^n V_n$. We will also make use of the following definitions:

$$\begin{aligned} \widetilde{V}^-_m &= \tfrac{\mathrm{i}}{2}\omega^*_{mn}V^{n+} \\ \widetilde{V}^+_m &= -\tfrac{\mathrm{i}}{2}\omega_{mn}V^{n-} \ , \end{aligned} \tag{B.23}$$

for any real vector V_m. Let us also mention that in deriving the general solution to the Killing spinor equations, it will be useful to take the following relations into account:

$$\begin{aligned} \gamma_{mn}\eta &= i j_{mn}\eta + i\omega_{mn}\eta^c \\ \gamma_{mn}\eta^c &= -i j_{mn}\eta^c + i\omega^*_{mn}\eta \ , \end{aligned} \tag{B.24}$$

which can be shown by Fierzing; see [9, 26, 24, 25] for a more detailed discussion of $SU(2)$ structures.

The torsion classes of the $SU(2)$ structure of $T\mathcal{M}_4$ parameterize the failure of η to be covariantly constant. Explicitly, we define the torsion classes $\mathcal{W}^{(i)}_m$, $i = 1, 2$, via:

$$\nabla_m\eta = \mathcal{W}^{(1)}_m\eta + \mathcal{W}^{(2)}_m\eta^c \ , \tag{B.25}$$

where as in the IIA case, $\mathcal{W}^{(2)}$ is a complex one-form, and $\mathcal{W}^{(1)}$ is an imaginary one-form. Alternatively the torsion classes can be defined in terms of the exterior derivatives of j, ω. Indeed, from eq. (B.25) we have, upon taking definition (B.20) into account:

$$\begin{aligned} \mathrm{d}j &= \mathcal{W}^*_2 \wedge \omega + \mathcal{W}_2 \wedge \omega^* \\ \mathrm{d}\omega &= 2\mathcal{W}_1 \wedge \omega - 2\mathcal{W}_2 \wedge j \ . \end{aligned} \tag{B.26}$$

As already mentioned, the spinor η further reduces the structure group of $T\mathcal{M}_4$ from $SO(4) \cong SU(2) \times SU(2)'$ (which is accomplished by the existence of a Riemannian metric on $\mathcal{M}_4$) to $SU(2)$. The spinors η, η^c are singlets under the first $SU(2)$ factor, whereas they transform as an $SU(2)'$ doublet under the second factor. Moreover there is an alternative

$SU(2)'$-covariant description of the $SU(2)$ structure on $T\mathcal{M}_4$ and its associated torsion classes, which can be seen as follows:[12] Let us define a triplet of real two-forms j_i, and a triplet of real one-forms $\mathcal{W}_i$, $i = 1, 2, 3$, via

$$(j_1, j_2, j_3) := (j, \mathrm{Re}\omega, -\mathrm{Im}\omega) \ ; \quad (\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3) := (\mathrm{Im}\mathcal{W}^{(1)}, \mathrm{Im}\mathcal{W}^{(2)}, -\mathrm{Re}\mathcal{W}^{(2)}) \ . \tag{B.27}$$

It can be seen that the j_i's transform as a triplet of $SU(2)'$, and moreover eqs. (B.26) can be cast in an $SU(2)'$-covariant form:

$$\mathrm{d}j_m = 2\varepsilon_{mnp}\mathcal{W}_n \wedge j_p \ . \tag{B.28}$$

We may use this $SU(2)'$ gauge freedom to rotate the torsion classes in eq. (B.26) to a more standard form, as in [9].

We now proceed by giving the most general ansatz for all forms. The (magnetic) RR flux F, *cf.* eq. (3.3), can be expanded as:

$$F = F_1 + F_3 \ , \tag{B.29}$$

with:

$$\begin{aligned} e^{\Phi}F_1 &= f^{(1)} \\ e^{\Phi}F_3 &= \star_4 f^{(3)} \ , \end{aligned} \tag{B.30}$$

where $f^{(1)}$, $f^{(3)}$, are real one-forms on $\mathcal{M}_4$. Similarly, we decompose the NSNS three-form as follows:

$$H = \star_4 h \ , \tag{B.31}$$

where h is a real one-form.

We are now ready to give the solution to the background supersymmetry equations, by plugging the above expansions into the Killing spinor equations (B.1), taking eq. (B.24) into account. The equality of the norms of $\eta_{1,2}$ imposes:

$$a^2 = |b|^2 + |c|^2 \ , \tag{B.32}$$

which we will assume to hold in the following; moreover, we will take $b \in \mathbb{R}$ for simplicity. Explicitly, the fluxes are given by:

$$\begin{aligned} f^{(1)}_m &= -\frac{4}{a}\left(c(\widetilde{\partial_m A})^- + c^*(\widetilde{\partial_m A})^+\right) \\ f^{(3)}_m &= \frac{4b}{a}(\partial_m A) \\ h_m &= -\frac{4b}{a^2}\left(c(\widetilde{\partial_m A})^- + c^*(\widetilde{\partial_m A})^+\right) \ . \end{aligned} \tag{B.33}$$

[12] The following two equations were worked out together with Diederik Roest.

The torsion classes read:

$$\begin{aligned}
\mathcal{W}_m^{(1)+} &= \frac{2b^2-a^2}{2a^2}(\partial_m A)^+ \\
\mathcal{W}_m^{(1)-} &= -\frac{2b^2-a^2}{2a^2}(\partial_m A)^- \\
\\
\mathcal{W}_m^{(2)+} &= \frac{b}{a^2}\left(c(\partial_m A)^+ + b(\widetilde{\partial_m A})^+\right) \\
\mathcal{W}_m^{(2)-} &= \frac{c}{a^2}\left(c(\widetilde{\partial_m A})^- + b(\partial_m A)^-\right)\ .
\end{aligned} \tag{B.34}$$

Moreover, we have:

$$\begin{aligned}
(\partial_m a) &= \frac{a}{2}(\partial_m A) \\
(\partial_m b) &= \frac{1}{2}b\frac{5a^2-4b^2}{a^2}(\partial_m A) \\
(\partial_m c) &= c\frac{a^2-4b^2}{2a^2}(\partial_m A)
\end{aligned} \tag{B.35}$$

and

$$(\partial_m \Phi) = \frac{2(a^2+|c|^2)}{a^2}(\partial_m A)^-\ . \tag{B.36}$$

To make contact with the supersymmetry equations in pure-spinor form (4.11), we note that the definition (2.2) implies:

$$\begin{aligned}
\Psi_1 &= \frac{1}{a}\left(b - b\mathrm{vol}_4 - \mathrm{i}(bj + c^*\omega)\right) \\
\Psi_2 &= \frac{1}{a}\left(c - c\mathrm{vol}_4 + \mathrm{i}(b\omega - cj)\right)\ ,
\end{aligned} \tag{B.37}$$

where vol_4 is the volume form of $\mathcal{M}_4$, and we have taken eqs. (B.19), (B.20), (B.24) into account. It is then straightforward to show that the solution of the Killing spinor equations given above also solves eqs. (4.11), upon taking (B.37), (B.26), (B.30), (B.31) into account.

C. Explicit examples

As an illustration of the pure-spinor formalism in the $d = 6$ case, we will now construct a type IIB warped K3 solution with spacetime-filling D5 branes localized on the K3. We also construct a IIA warped $S^1 \times T^3$ solution with spacetime-filling D6 branes localized on the T^3 and wrapping the S^1. The two solutions are related by T-duality, in the case where on the IIB side the K3 is replaced by a T^4.

The IIB solution

The ten-dimensional metric is of the form:

$$\mathrm{d}s^2 = e^{2A}\mathrm{d}s^2(\mathbb{R}^{1,5}) + e^{-2A}\mathrm{d}s^2(\mathrm{K3})\ . \tag{C.1}$$

Correspondingly, the $SU(2)$ structure (j, ω) obeys:

$$\mathrm{d}(e^{2A} j) = \mathrm{d}(e^{2A} \omega) = 0 \ . \tag{C.2}$$

Taking the equations above into account, it can be seen that the pure spinors of eq. (B.37) solve the supersymmetry equations (4.11), provided the remaining fields are given by:

$$\begin{aligned} F_1 &= 0 \\ F_3 &= 4e^{-2A} \star_4 \mathrm{d}A \\ H &= 0 \\ \Phi &= 2A \ , \end{aligned} \tag{C.3}$$

and we also set: $a = b$, $c = 0$. In order to have a solution to the full set of equations of motion, it suffices to impose in addition the Bianchi equations for all fields [21, 22, 13]. It is not difficult to see that this leads to one additional equation:

$$\nabla^2_{\mathrm{K3}} e^{-4A} = 0 \ , \tag{C.4}$$

i.e. e^{-4A} is harmonic with respect to the K3 metric.[13] The solution also admits spacetime-filling D5 branes localized on K3 (as can be seen from the form of the RR three-form flux in the solution above), which can be introduced by replacing the right-hand side above with a delta function on K3.

Upon replacing the K3 by a T^4, the solution coincides with the one obtained using the 'harmonic superposition rules' for a stack of D5 branes in flat space (see [27] for a review). Moreover, one can 'smear' the warp factor A along one direction of the torus (*i.e.* assume that A is independent of the corresponding coordinate) and T-dualize to IIA along the smeared direction. The T-dual is a warped $S^1 \times T^3$ solution with spacetime-filling D6 branes localized on the T^3 and wrapping the S^1.

The IIA solution

We would now like to describe the T-dual warped $S^1 \times T^3$ solution with spacetime-filling D6 branes, mentioned in the previous subsection, in the language of pure spinors.

The ten-dimensional metric is of the form:

$$ds^2 = e^{2A} \left(\mathrm{d}s^2(\mathbb{R}^{1,5}) + \mathrm{d}\lambda^2 \right) + e^{-2A} \mathrm{d}s^2(T^3) \ , \tag{C.5}$$

where the coordinate λ parameterizes an S^1. Correspondingly, the complex one-forms u, v may be chosen as follows:

$$u = e^{-A} \left(\mathrm{d}y_1^2 + \mathrm{i}\mathrm{d}y_2^2 \right) \ ; \quad v = e^{A} \mathrm{d}\lambda^2 + \mathrm{i}e^{-A} \mathrm{d}y_3^2 \ , \tag{C.6}$$

[13] A harmonic function on a smooth Riemannian manifold without boundary is constant; in our case this would lead to a constant warp factor, and all flux would vanish.

where y_1, y_2, y_3 are coordinates of T^3 such that $\mathrm{d}s^2(T^3) = \mathrm{d}y_i \mathrm{d}y_i$. Taking the equations above into account, it can be seen that the pure spinors of eq. (B.18) solve the supersymmetry equations (4.11), provided the remaining fields are given by:

$$
\begin{aligned}
F_0 &= 0 \\
F_2 &= -4e^{-2A} \star_4 (\mathrm{d}A \wedge \mathrm{d}\lambda) \\
F_4 &= 0 \\
H &= 0 \\
\Phi &= 3A \ .
\end{aligned}
\tag{C.7}
$$

In order to have a solution to the full set of equations of motion, it suffices to impose in addition the Bianchi equations for all fields [21, 22, 13]. It is not difficult to see that this leads to one additional equation:

$$
\nabla^2_{\mathrm{T}^3} e^{-4A} = 0 \ , \tag{C.8}
$$

i.e. e^{-4A} is harmonic with respect to the metric on T^3 (*cf.* the last footnote). The solution also admits spacetime-filling D6 branes localized on T^3 and wrapping the S^1 parameterized by λ (as can be seen from the form of the RR two-form flux in the solution above), which can be introduced by replacing the right-hand side above with a delta function on T^3.

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